Note on the classification of holonomy algebras of Lorentzian manifolds

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Abstract

The classification of the holonomy algebras of Lorentzian manifolds can be reduced to the classification of irreducible subalgebras $\mathfrak{h}\subset\mathfrak{so}(n)$ that are spanned by the images of linear maps from $\mathbb{R}^n$ to $\mathfrak{h}$ satisfying an identity similar to the Bianchi one. T. Leistner found all such subalgebras and it turned out that the obtained list coincides with the list of irreducible holonomy algebras of Riemannian manifolds. The natural problem is to give a simple direct proof to this fact. We give such proof for the case of semisimple not simple Lie algebras $\mathfrak{h}$.

1 Introduction

M. Berger [2] classified possible connected irreducible holonomy groups $H\subset \mathrm{SO}(n)$ of not locally symmetric Riemannian manifolds using the representation theory. It turned out that these groups act transitively on the unite sphere of the tangent space. J. Simens [10] and recently in a simple geometric way C. Olmos [9] proved this result directly.

The classification of the holonomy algebras (i.e. the Lie algebras of the holonomy groups) of Lorentzian manifolds can be reduced to the classification of irreducible weak-Berger subalgebras $\mathfrak{h}\subset\mathfrak{so}(n)$, i.e. subalgebras $\mathfrak{h}\subset\mathfrak{so}(n)$ that are spanned by the images of linear maps from the space

$$\mathcal{P}(\mathfrak{h})=\{P\in(\mathbb{R}^n)^*\otimes\mathfrak{h}|(P(u)v,w)+(P(v)w,u)+(P(w)u,v)=0 \text{ for all } u,v,w\in\mathbb{R}^n\}.$$

It is easy to see that if $\mathfrak{h}\subset\mathfrak{so}(n)$ is the holonomy algebras of a Riemannian manifold, then it is a weak-Berger algebra. The inverse statement is absolutely not obvious, nevertheless it is true and it is proven by T. Leistner in [8].

If n is even and $\mathfrak{h}\subset\mathfrak{so}(n)$ is of complex type, i.e. $\mathfrak{h}\subset\mathfrak{u}(\frac{n}{2})$, then it can be shown that $\mathcal{P}(\mathfrak{h})\simeq(\mathfrak{h}\otimes\mathbb{C})^{(1)}$, where $(\mathfrak{h}\otimes\mathbb{C})^{(1)}$ is the first prolongation of the subalgebra $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{gl}(n,\mathbb{C})$ (cf. [8] and [6]). Using that and the classification of irreducible representations with non-trivial prolongation, Leistner showed that if $\mathfrak{h}\subset\mathfrak{u}(\frac{n}{2})$ is a weak-Berger subalgebra, then it is the holonomy algebra of a Riemannian manifold.

The situation when $\mathfrak{h}\subset\mathfrak{so}(n)$ is of real type (i.e. not of complex type) is much more difficult. In this case Leistner considered the complexification $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{so}(n,\mathbb{C})$, which is irreducible. He used the classification of irreducible representations of complex semisimple Lie algebras, found a criteria in terms of weights for such representation $\mathfrak{h}\otimes\mathbb{C}\subset\mathfrak{so}(n,\mathbb{C})$ to be a weak-Berger algebra and considered case by case simple Lie algebras $\mathfrak{h}\otimes\mathbb{C}$, and then semisimple Lie algebras (the problem is reduced to the semisimple Lie algebras of the form $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{k}$, where $\mathfrak{k}$ is simple, and again different possibilities for $\mathfrak{k}$ were considered).

We consider the case of semisimple not simple irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ with irreducible complexification $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$. In a simple way we show that it is enough to treat the case when $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}$, where $\mathfrak{k} \subsetneq \mathfrak{sp}(2m, \mathbb{C})$ is a proper irreducible subalgebra, and the representation space is the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^{2m}$. We show that in this case $\mathcal{P}(\mathfrak{h})$ coincides with the first Tanaka prolongation $\mathfrak{g}_1$ of the non-negatively graded Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

where $\mathfrak{g}_{-2} = \mathbb{C}, \quad \mathfrak{g}_{-1} = \mathbb{C}^{2m}, \quad \mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}\,\mathrm{id}_{\mathbb{C}^{2m}}$, and the grading is defined by the element $-\mathrm{id}_{\mathbb{C}^{2m}}$. We prove that if $\mathcal{P}(\mathfrak{h})$ is non-zero, then $\mathfrak{g}_1$ is isomorphic to $\mathbb{C}^{2m}$, the second Tanaka prolongation $\mathfrak{g}_2$ is isomorphic to $\mathbb{C}$, and $\mathfrak{g}_3 = 0$. Then, the full Tanaka prolongation defines the simple $|2|$-graded complex Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

It is well known that such Lie algebra defines (up the duality) a simply connected Riemannian symmetric space; the holonomy algebra of this space coincides with $\mathfrak{h} \subset \mathfrak{so}(n)$. Thus, if the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is semisimple and not simple, and $\mathcal{P}(\mathfrak{h}) \neq 0$, then we indicate a Riemannian manifold with the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$.

More details about the holonomy algebras of Lorentzian manifolds can be found in [4, 5].

2 Holonomy algebras of Riemannian manifolds

Irreducible holonomy algebras $\mathfrak{h} \subset \mathfrak{so}(n)$ of not locally symmetric Riemannian manifolds are exhausted by $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$, $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$, $\mathfrak{sp}(\frac{n}{4})$, $G_2 \subset \mathfrak{so}(7)$ and $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$. This list (up to some corrections) obtained M. Berger [2]. Berger classified irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ spanned by the images of the maps from the space

$$\mathcal{R}(\mathfrak{h}) = \{R \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{h} | R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \text{ for all } X, Y, Z \in \mathbb{R}^n\}$$

of algebraic curvature tensors of type $\mathfrak{h}$ under the condition that the space

$$\mathcal{R}^\nabla(\mathfrak{h}) = \{S \in (\mathbb{R}^n)^* \otimes \mathcal{R}(\mathfrak{h}) | S_X(Y,Z) + S_Y(Z,X) + S_Z(X,Y) = 0 \text{ for all } X, Y, Z \in \mathbb{R}^n\}$$

of algebraic covariant derivatives of the curvature tensors of type $\mathfrak{h}$ is not trivial. Berger used the classification of irreducible representations of compact Lie groups. The connected Lie subgroups of $\mathrm{SO}(n)$ corresponding to the above subalgebras of $\mathfrak{so}(n)$ mostly exhaust groups of isometries acting transitively on the unite sphere of dimension $n-1$, and the result of Berger can be reformulated in the following form: if the irreducible holonomy group of a Riemannian manifold (M, g) does not act transitively on the unite sphere of the tangent space, then (M, g) is locally symmetric. A direct proof of this statement obtained in algebraic way J. Simens [10], and recently an elegant geometric proof obtained C. Olmos [9].

The spaces $\mathcal{R}(\mathfrak{h})$ for the holonomy algebras of Riemannian manifolds $\mathfrak{h} \subset \mathfrak{so}(n)$ are computed by D. V. Alekseevsky in [1]. For $R \in \mathcal{R}(\mathfrak{h})$ define its Ricci tensor by

$$\mathrm{Ric}(R)(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y),$$

$X, Y \in \mathbb{R}^n$. Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible Riemannian holonomy algebra. The space $\mathcal{R}(\mathfrak{h})$ admits the following decomposition into $\mathfrak{h}$-modules

$$\mathcal{R}(\mathfrak{h}) = \mathcal{R}_0(\mathfrak{h}) \oplus \mathcal{R}_1(\mathfrak{h}) \oplus \mathcal{R}'(\mathfrak{h}), \tag{1}$$

where $\mathcal{R}_0(\mathfrak{h})$ consists of the curvature tensors with zero Ricci curvature, $\mathcal{R}_1(\mathfrak{h})$ consists of tensors annihilated by $\mathfrak{h}$ (this space is zero or one-dimensional), $\mathcal{R}'(\mathfrak{h})$ is the complement to

these two spaces. If $\mathcal{R}(\mathfrak{h}) = \mathcal{R}_1(\mathfrak{h})$, then any Riemannian manifold with the holonomy algebra $\mathfrak{h}$ is locally symmetric. Such subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ is called *a symmetric Berger algebra*. The holonomy algebras of irreducible Riemannian symmetric spaces are exhausted by $\mathfrak{so}(n)$, $u(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$ and by symmetric Berger algebras $\mathfrak{h} \subset \mathfrak{so}(n)$.

It is known that connected and simply connected indecomposable symmetric Riemannian manifolds (M, g) are in one-two-one correspondence with simple $\mathbb{Z}_2$-graded Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ such that $\mathfrak{h} \subset \mathfrak{so}(n)$. The subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ coincides with the holonomy algebra of (M, g). The space (M, g) can be reconstructed using its holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ and the value $R \in \mathcal{R}(\mathfrak{h})$ of the curvature tensor of (M, g) at a point. Define the structure of the Lie algebra on the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}^n$ in the following way:

$$[A, B] = [A, B]_{\mathfrak{h}}, \quad [A, X] = AX, \quad [X, Y] = R(X, Y), \quad A, B \in \mathfrak{h},\ X, Y \in \mathbb{R}^n.$$

Then $M = G/H$, where G is the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$, and $H \subset G$ is the connected Lie subgroup corresponding to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

In the case of Hermitian symmetric spaces it holds $\mathfrak{h} \subset \mathfrak{u}(m)$, $n = 2m$. The complexification of the Lie algebra $\mathfrak{h} \oplus \mathbb{R}^{2m}$ gives the simple $\mathbb{Z}$-graded Lie algebra of depth one

$$\mathbb{C}^m \oplus (\mathfrak{h} \otimes \mathbb{C}) \oplus (\mathbb{C}^m)^*.$$

Conversely, any complex simple $\mathbb{Z}$-graded Lie algebra of the form

$$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = V \oplus \mathfrak{g}_0 \oplus V^*$$

defines up to the duality a simply connected Hermitian symmetric space.

If the symmetric space is quaternionic-Kählerian, then $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{f} \subset \mathfrak{so}(4k)$, where $n = 4k$ and $\mathfrak{f} \subset \mathfrak{sp}(k)$. The complexification of $\mathfrak{h} \oplus \mathbb{R}^{4k}$ is equal to $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2k})$, where $\mathfrak{k} = \mathfrak{f} \otimes \mathbb{C} \subset \mathfrak{sp}(2k, \mathbb{C})$. Let e_1, e_2 be the basis of $\mathbb{C}^2$, and let

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the standard basis of $\mathfrak{sl}(2, \mathbb{C})$. We obtain the following $\mathbb{Z}$-grading of $\mathfrak{g} \otimes \mathbb{C}$:

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathbb{C}F \oplus e_2 \otimes \mathbb{C}^{2k} \oplus (\mathfrak{k} \oplus \mathbb{C}H) \oplus e_1 \otimes \mathbb{C}^{2k} \oplus \mathbb{C}E.$$

Conversely, any such simple $\mathbb{Z}$-graded Lie algebra defines up to the duality a simply connected quaternionic-Kählerian symmetric space.

3 Weak curvature tensors

The spaces $\mathcal{P}(\mathfrak{h})$ are computed in [6]. Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible subalgebra. There exists the decomposition

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}),$$

where $\mathcal{P}_0(\mathfrak{h})$ is the kernel of the $\mathfrak{h}$-equivariant map

$$\widetilde{\mathrm{Ric}} : \mathcal{P}(\mathfrak{h}) \to \mathbb{R}^n, \qquad \widetilde{\mathrm{Ric}}(P) = \sum_{i=1}^{n} P(e_i)e_i$$

$(e_1, ..., e_n$ is an orthogonal basis of $\mathbb{R}^n)$, and $\mathcal{P}_1(\mathfrak{h})$ is the orthogonal complement of $\mathcal{P}_0(\mathfrak{h})$ in $\mathcal{P}(\mathfrak{h})$. The space $\mathcal{P}_1(\mathfrak{h})$ is either trivial or it is isomorphic to $\mathbb{R}^n$. It holds $\mathcal{P}_0(\mathfrak{h}) \neq 0$ if and only if $\mathcal{R}_0(\mathfrak{h}) \neq 0$. Next, $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ if and only if $\mathcal{R}_1(\mathfrak{h}) \simeq \mathbb{R}$. For the holonomy algebras of symmetric Riemannian spaces different from $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$ and $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1))$ it holds $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$ and $\mathcal{P}_0(\mathfrak{h}) = 0$.

4 Tanaka prolongations

Consider a $\mathbb{Z}$-graded Lie algebra of the form

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

For $k \geq 1$, the k-th Tanaka prolongation is defined by the induction

$$\mathfrak{g}_k = \{u \in (\mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{k-2}) \oplus (\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{k-1}) | u([X,Y]) = [u(X),Y] + [X,u(Y)],\ X,Y \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}\}.$$

Let $k \geq 1$, and $l \geq 0$. For $u \in \mathfrak{g}_k$ and $v \in \mathfrak{g}_l$ define Lie brackets $[u,v] \in \mathfrak{g}_{k+l}$ by the condition

$$[u,v]X = [[u,X],v] + [u,[v,X]], \quad X \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1};$$

the Lie brackets of $u \in \mathfrak{g}_k$ and $X \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ are defined as $[u,X] = Xu$. This gives the structure of a Lie algebra on the vector space $\oplus_{k=-2}^{\infty} \mathfrak{g}_k$.
Let $\mathfrak{k} \subset \mathfrak{sp}(2m,\mathbb{C})$ be a subalgebra, $m \geq 2$. Consider the Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0, \quad \mathfrak{g}_{-2} = \mathbb{C}F, \quad \mathfrak{g}_{-1} = \mathbb{C}^{2m}, \quad \mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}H$$

with the non-zero Lie brackets

$$[X,Y] = \Omega(X,Y)F, \quad [A,X] = AX, \quad [A,B] = [A,B]_{\mathfrak{k}}, \quad [H,X] = -X, \quad [H,F] = -2F,$$

where $X, Y \in \mathbb{C}^{2m}$, $A, B \in \mathfrak{k}$, and Ω is the symplectic form on $\mathbb{C}^{2m}$.

Lemma 1 *It holds*

$$\mathfrak{g}_1 = \{\varphi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 | \exists A \in \mathfrak{g}_{-1},\ \varphi(X)Y - \varphi(Y)X = \Omega(X,Y)A,\ X,Y \in \mathfrak{g}_{-1}\}.$$

If $\mathfrak{k} \subsetneq \mathfrak{sp}(2m,\mathbb{C})$ *is a proper irreducible subalgebra and* $\mathfrak{g}_1 \neq 0$, *then* $\mathfrak{g}_1 \simeq \mathbb{C}^{2m}$, $\mathfrak{g}_2 \simeq \mathbb{C}$, *and* $\mathfrak{g}_3 = 0$. *The Lie algebra*

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

is simple.

Proof. Let $u = \psi + \varphi$, where $\psi \in \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1}$, and $\varphi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$. The condition $u \in \mathfrak{g}_1$ is equivalent to the equations

$$[\varphi(X), F] = \Omega(\psi(F), Y)F, \quad \varphi(X)Y - \varphi(Y)X = \Omega(X,Y)\psi(F).$$

The first statement of the lemma is that the second equation implies the first one.
Let us denote $\mathbb{C}^{2m}$ by V. First suppose that $\mathfrak{k} = \mathfrak{sp}(V)$. Let us find $\mathfrak{g}_1$. As the $\mathfrak{sp}(V)$-modules, we have $\mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1} \simeq V$, and

$$\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \simeq V \otimes (\mathfrak{sp}(V) \oplus \mathbb{C}) = V \oplus (V \oplus V_{3\pi_1} \oplus V_{\pi_1+\pi_2}),$$

where V_Λ denotes the irreducible $\mathfrak{sp}(V)$-module with the highest weight Λ. By the definition, the intersection of $\mathfrak{g}_1$ and $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ coincides with

$$(\mathfrak{sp}(V) \oplus \mathbb{C}H)^{(1)} = (\mathfrak{sp}(V))^{(1)} = \odot^3 V \simeq V_{3\pi_1}.$$

Clearly, the intersection of $\mathfrak{g}_1$ and $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ is trivial. Consequently, if $\mathfrak{g}_1$ is different from $\mathfrak{sp}(V)^{(1)}$, then $\mathfrak{g}_1$ contains a submodule isomorphic to V. Any $\mathfrak{sp}(V)$-equivariant map from V to $(\mathfrak{g}^*_{-2} \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0)$ is of the form

$$Z \mapsto \psi^Z + \varphi^Z, \quad \psi^Z(F) = aZ, \quad \varphi^Z(Y) = b\Omega(Z,Y)H + cZ \odot Y,$$

where $a, b, c \in \mathbb{R}$, and $Z \odot Y \in \mathfrak{sp}(V)$ is defined as

$$(Z \odot Y)X = \Omega(Z, X)Y + \Omega(Y, X)Z.$$

The second equation on $\mathfrak{g}_1$ takes the form

$$-b\Omega(Z, X)Y + b\Omega(Z, Y)X + c(\Omega(Y, Z)X - \Omega(X, Z)Y + 2\Omega(Y, X)Z) = a\Omega(X, Y)Z.$$

This equation should hold for all $X, Y, Z \in V$, and it is equivalent to $b = -c = -\frac{1}{2}a$ (since $\dim V \geq 4$). The second equation on $\mathfrak{g}_1$ takes the form

$$-2b\Omega(Z, Y) = a\Omega(Z, Y)$$

and it follows from the first one. Thus the orthogonal complement to $(\mathfrak{sp}(2m, \mathbb{C}))^{(1)}$ in $\mathfrak{g}_1$ is isomorphic to V, and the isomorphism is given by

$$Z \in V \mapsto \psi^Z + \varphi^Z, \quad \psi^Z(F) = 2Z, \quad \varphi^Z(Y) = -\Omega(Z, Y)H + Z \odot Y, \quad Y \in V.$$

Let $\mathfrak{k} \subsetneq \mathfrak{sp}(V)$ be a proper irreducible subalgebra. It is clear that

$$\mathfrak{g}_1 = ((\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}) \oplus (\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0)) \cap (\mathfrak{sp}(V) \oplus \mathbb{C}H)_1,$$

and $\mathfrak{h}^{(1)} = (\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0) \cap \mathfrak{sp}(V)^{(1)}$. It is known that $\mathfrak{h}^{(1)} = 0$. Consequently, if $\mathfrak{g}_1 \neq 0$, than it is isomorphic to V and it is included diagonally into $V \oplus \mathfrak{sp}(V)^{(1)}$.

Consider the full Tanaka prolongation $\mathfrak{g} = \oplus_{i=-2}^{\infty}\mathfrak{g}_i$. Let $\mathfrak{g}^0 = \oplus_{i=0}^{\infty}\mathfrak{g}_i \subset \mathfrak{g}$. We claim that $\mathfrak{g}$ is a primitive $\mathbb{Z}$-graded Lie algebra, i.e. $\mathfrak{g}^0 \subset \mathfrak{g}$ is a maximal graded subalgebra and $\mathfrak{g}^0$ contains no graded ideals of $\mathfrak{g}$ except $\{0\}$. Indeed, suppose that there exists a subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ such that $\mathfrak{g}^0 \subsetneq \tilde{\mathfrak{g}}$. Then $aF + X \in \tilde{\mathfrak{g}}$ for some $a \in \mathbb{R}$, $X \in \mathfrak{g}_{-1}$. If $a \neq 0$, then taking $u \in \mathfrak{g}_1$, we get $0 \neq u(F) \in \tilde{\mathfrak{g}} \cap \mathfrak{g}_{-1}$, i.e. we may assume that there exists non-zero $X \in \mathfrak{g}_{-1}$ such that $X \in \tilde{\mathfrak{g}}$. Since $\mathfrak{g}_0$ acts on $\mathfrak{g}_{-1}$ irreducible, we get $\mathfrak{g}_{-1} \subset \tilde{\mathfrak{g}}$. Finally, $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, i.e. $\mathfrak{g}_{-2} \subset \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \mathfrak{g}$. Suppose now that $\tilde{\mathfrak{g}} = \oplus_{i=0}^{\infty}\tilde{\mathfrak{g}}_i \subset \mathfrak{g}^0$ is a graded ideal. For $X \in \mathfrak{g}_{-1}$ and $\xi \in \tilde{\mathfrak{g}}_0$ it holds $[\xi, X] \in \mathfrak{g}_{-1}$. On the other hand, $[\xi, X] \in \tilde{\mathfrak{g}}$, and we get $[\xi, X] = 0$ for all $X \in \mathfrak{g}_{-1}$. This implies $\tilde{\mathfrak{g}}_0 = 0$. In the same way it can be shown that $\tilde{\mathfrak{g}}_k = 0$ for all $k \geq 2$. Thus, $\mathfrak{g}$ is a primitive $\mathbb{Z}$-graded Lie algebra. If $\mathfrak{g}$ is infinite dimensional, then from [7, Th. 6.1] it follows that $\mathfrak{g}_0 = \mathfrak{sp}(V) \oplus \mathbb{C}H$, which gives a contradiction, since we assume that $\mathfrak{k} \subsetneq \mathfrak{sp}(V)$ is a proper subalgebra. Thus, $\mathfrak{g}$ is of finite dimension. Since the element $H \in \mathfrak{g}_0$ defines the $\mathbb{Z}$-grading of $\mathfrak{g}$, any ideal $\mathfrak{t} \subset \mathfrak{g}$ is graded. As in the above claim it can be shown that either $\mathfrak{t} = \mathfrak{g}$ or $\mathfrak{t} = 0$, i.e. $\mathfrak{g}$ is a simple Lie algebra. For the Killing form of a $\mathbb{Z}$-graded Lie algebra it holds $b(\mathfrak{g}_k, \mathfrak{g}_l) = 0$, unless $k = -l$. This shows that $\mathfrak{g}_2 \simeq \mathbb{C}$ and $\mathfrak{g}_3 = 0$. The lemma is proved. $\Box$

5 Semisimple not simple weak-Berger algebras

Theorem 1 *Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be a semisimple not simple irreducible subalgebra of real type. If $\mathcal{P}(\mathfrak{h}) \neq 0$, then $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a symmetric Riemannian space.*

Proof. From the assumption of the theorem it follows that the complexified representation $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ is irreducible. Since $\mathfrak{h} \otimes \mathbb{C}$ is semisimple and not simple, it can be written as the sum of two ideals, $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. The representation of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ on $\mathbb{C}^n$ must be of the from of the tensor product, $\mathbb{C}^n = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$, where $\mathfrak{h}_1 \subset \mathfrak{gl}(n_1, \mathbb{C})$, $\mathfrak{h}_2 \subset \mathfrak{gl}(n_2, \mathbb{C})$ are irreducible. Since $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{so}(n, \mathbb{C})$, it holds either $\mathfrak{h}_1 \subset \mathfrak{so}(n_1, \mathbb{C})$, $\mathfrak{h}_2 \subset \mathfrak{so}(n_2, \mathbb{C})$, $n_1, n_2 \geq 3$ or $\mathfrak{h}_1 \subset \mathfrak{sp}(n_1, \mathbb{C})$, $\mathfrak{h}_2 \subset \mathfrak{sp}(n_2, \mathbb{C})$, $n_1, n_2 \geq 2$. In [6] it is shown in a simple way that $\mathcal{P}(\mathfrak{so}(n_1, \mathbb{C}) \oplus \mathfrak{so}(n_2, \mathbb{C})) \simeq \mathbb{C}^n$, and if $n_1, n_2 \geq 3$, then $\mathcal{P}(\mathfrak{sp}(n_1, \mathbb{C}) \oplus \mathfrak{sp}(n_2, \mathbb{C})) \simeq \mathbb{C}^n$.

This implies that if $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a proper irreducible subalgebra of $\mathfrak{so}(n_1,\mathbb{C}) \oplus \mathfrak{so}(n_2,\mathbb{C})$ or of $\mathfrak{sp}(n_1,\mathbb{C}) \oplus \mathfrak{sp}(n_2,\mathbb{C})$ with $n_1, n_2 \geq 3$, then $\mathcal{P}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) = 0$. Thus, we are left with the case $n_1 = 2$, $\mathfrak{h}_1 = \mathfrak{sl}(2,\mathbb{C})$. Let $\mathfrak{k} = \mathfrak{h}_2$. If $\mathfrak{k} = \mathfrak{sp}(n_2,\mathbb{C})$, then $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(k) \subset \mathfrak{so}(4k)$ is the holonomy algebra of a quaternionic-Kählerian manifold ($n = 4k = 2n_2$). Suppose that $\mathfrak{k} \subsetneq \mathfrak{sp}(n_2,\mathbb{C})$. From Proposition 1 below, Lemma 1, and the considerations of Section 2 it follows that $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{f} \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(k) \subset \mathfrak{so}(4k)$ is the holonomy algebra of a quaternionic-Kählerian symmetric space. □

Proposition 1 *Let $\mathfrak{k} \subset \mathfrak{sp}(2m,\mathbb{C})$ be an irreducible subalgebra, $m \geq 2$. Then*

$$\mathcal{P}(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k}) \simeq \mathbb{C}^2 \otimes \mathfrak{g}_1,$$

where $\mathfrak{g}_1$ is the first Tanaka prolongation of the Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \mathbb{C}F \oplus \mathbb{C}^{2m} \oplus (\mathfrak{k} \oplus \mathbb{C}H).$$

Proof. Let $V = \mathbb{C}^{2m}$, let Ω, ω be the symplectic forms on V and $\mathbb{C}^2$, and let e_1, e_2 be a basis of $\mathbb{C}^2$ such that $\omega(e_1, e_2) = 1$. Let F, H, E be the basis of $\mathfrak{sl}(2,\mathbb{C})$ as above. For a linear map $P : \mathbb{C}^2 \otimes V \to \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k}$ and $X \in V$ we write

$$P(e_i \otimes X) = \alpha(e_i \otimes X)E + \beta(e_i \otimes X)F + \gamma(e_i \otimes X)H + T(e_i \otimes X), \quad T(e_i \otimes X) \in \mathfrak{k}, \quad i = 1, 2.$$

Let us consider the condition $P \in \mathcal{P}(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k})$. Let $X, Y, Z \in V$. Taking the vectors $e_1 \otimes X$, $e_1 \otimes Y$, $e_1 \otimes Z$, we get

$$\beta(e_1 \otimes X)\Omega(Y, Z) + \beta(e_1 \otimes Y)\Omega(Z, X) + \beta(e_1 \otimes Z)\Omega(X, Y) = 0.$$

Since $\dim V \geq 4$, this implies $\beta(e_1 \otimes X) = 0$ for all $X \in V$. Similarly, considering the vectors $e_2 \otimes X$, $e_2 \otimes Y$, $e_2 \otimes Z$, we get $\alpha(e_2 \otimes X) = 0$.
Considering the vectors $e_1 \otimes X$, $e_1 \otimes Y$, $e_2 \otimes Z$, we obtain

$$\gamma(e_1 \otimes X)\Omega(Y, Z) + \Omega(T(e_1 \otimes X)Y, Z) - \gamma(e_1 \otimes Y)\Omega(X, Z) - \Omega(T(e_1 \otimes Y)X, Z) - \beta(e_2 \otimes Z)\Omega(Y, X) = 0.$$

Let $A \in V$ be the dual vector to $\beta|_{e_2 \otimes V}$, i.e. $\beta(e_2 \otimes Z) = \Omega(A, Z)$ for all $Z \in V$. We obtain

$$\gamma(e_1 \otimes X)Y + T(e_1 \otimes X)Y - \gamma(e_1 \otimes Y)X - T(e_1 \otimes Y)X + \Omega(X, Y)A = 0.$$

The last equation on P can be obtained in the same way and it is of the form

$$\gamma(e_2 \otimes X)Y - T(e_2 \otimes X)Y - \gamma(e_2 \otimes Y)X + T(e_2 \otimes Y)X + \Omega(X, Y)B = 0,$$

where $B \in V$ is defined by $\beta(e_1 \otimes Z) = \Omega(B, Z)$, $Z \in V$. We conclude that $P \in \mathcal{P}(\mathfrak{sl}(2,\mathbb{C})$ if and only if $\gamma(e_1 \otimes \cdot)H + T(e_1 \otimes \cdot)$ and $\gamma(e_2 \otimes \cdot)H - T(e_2 \otimes \cdot)$ belong to $\mathfrak{g}_1$. Thus, $P \in \mathcal{P}(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k}) \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathfrak{g}_1$, which is an isomorphism of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k}$-modules. □

6 Further remarks

We are left with the problem to give a direct proof of the fact that if for a real irreducible representation $\mathfrak{h} \subset \mathfrak{so}(n)$ of a simple Lie algebra $\mathfrak{h}$ it holds $\mathcal{P}(\mathfrak{h}) \neq 0$, then $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold. The following two cases should be considered: $\mathcal{P}_0(\mathfrak{h}) \neq 0$ and $\mathcal{P}_1(\mathfrak{h}) \neq 0$. It is necessary to prove that the first condition implies that $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a not locally symmetric Riemannian manifold; the second condition

implies that $\mathfrak{h} \subset \mathfrak{so}(n)$ may appear as the holonomy algebra of a symmetric Riemannian manifold. It would be useful to give a direct proof to the following statement:

If the connected Lie subgroup $H \subset \mathrm{SO}(n)$ corresponding to an irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ does not act transitively on the unite sphere, then $\mathcal{P}_0(\mathfrak{h}) = 0$.

The relation of the spaces $\mathcal{P}(\mathfrak{h})$ and $\mathcal{R}(\mathfrak{h})$ is the following

$$\mathcal{R}(\mathfrak{h}) = \{S \in \mathbb{R}^{n*} \otimes \mathcal{P}(\mathfrak{h}) | S(X)(Y) = -S(Y)(X)\}.$$

Consider the natural map

$$\tau : \mathbb{R}^n \otimes \mathcal{R}(\mathfrak{h}) \to \mathcal{P}(\mathfrak{h}), \quad \tau(X \otimes R) = R(X, \cdot) \in \mathcal{P}(\mathfrak{h}).$$

Using the results of [8], in [6] it is show that $\tau(\mathbb{R}^n \otimes \mathcal{R}_0(\mathfrak{h})) = \mathcal{P}_0(\mathfrak{h})$ (if $n \geq 4$) and $\tau(\mathbb{R}^n \otimes \mathcal{R}_1(\mathfrak{h})) = \mathcal{P}_1(\mathfrak{h})$. It would be useful to get a direct proof of these statements for each irreducible subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$.

Suppose that $\mathcal{P}_1(\mathfrak{h}) \neq 0$, i.e. $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$. Then there exists an $\mathfrak{h}$-equivariant linear isomorphism $S : \mathbb{R}^n \to \mathcal{P}_1(\mathfrak{h})$ defined up to a constant. It should be proved that $S(X)(Y) = -S(Y)(X)$, i.e. $S \in \mathcal{R}_1(\mathfrak{h})$.

The space $\mathcal{P}(\mathfrak{h})$ is contained in the tensor product $\mathbb{R}^n \otimes \mathfrak{h}$. A statement form [6] implies that the decomposition of $\mathbb{R}^n \otimes \mathfrak{h}$ into the sum of irreducible $\mathfrak{h}$-modules is of the form

$$\mathbb{R}^n \otimes \mathfrak{h} = k\mathbb{R}^n \oplus (\oplus_\lambda V_\lambda),$$

where k is the number of non-zero labels on the Dynkin diagram for the representation of $\mathfrak{h} \otimes \mathbb{C}$ on $\mathbb{C}^n$, and V_λ are pairwise non-isomorphic irreducible $\mathfrak{h}$-modules that are not isomorphic to $\mathbb{R}^n$. If $\mathcal{P}_0(\mathfrak{h}) \neq 0$, then it coincides with the highest irreducible component in $\mathbb{R}^n \otimes \mathfrak{h}$.

The space $\mathcal{R}(\mathfrak{h})$ is contained in $\odot^2\mathfrak{h}$ [1]. If $\mathcal{R}_1(\mathfrak{h}) \neq 0$, then it is spanned by the map $\mathrm{id}_\mathfrak{h} \in \odot^2\mathfrak{h} \subset \wedge^2\mathbb{R}^n \otimes \mathfrak{h}$, note that $\mathrm{id}_\mathfrak{h}(X, Y) = \mathrm{pr}_\mathfrak{h}(X \wedge Y)$. Consequently, if $\mathcal{R}_1(\mathfrak{h}) \neq 0$, then $\mathcal{P}_1(\mathfrak{h}) = \tau(\mathbb{R}^n \otimes \mathrm{id}_\mathfrak{h}) = \{\mathrm{pr}_\mathfrak{h}(X \wedge \cdot) | X \in \mathbb{R}^n\}$. But it is not clear why if $\mathcal{P}_1(\mathfrak{h}) \neq 0$, then it should coincide with $\tau(X \otimes \mathrm{id}_\mathfrak{h})$ (such statement would imply $\mathcal{R}_1(\mathfrak{h}) \simeq \mathbb{R}$).

The statement of the following lemma can be checked directly.

Lemma 2 *Let $S : \mathbb{R}^n \to \mathcal{P}(\mathfrak{h})$ be a linear map. Consider the map $T : \wedge^2\mathbb{R}^n \to \mathfrak{h}$, $T(X, Y) = S(X)(Y) - S(Y)(X)$. Then $T + T^* \in \mathcal{R}(\mathfrak{so}(n))$, where $T^* : \mathfrak{so}(n) \to \mathfrak{so}(n)$ is given by $(T^*(X, Y)Z, W) = (T^*(Z, W)X, Y)$.*

We are able to show that the condition $\mathcal{P}_1(\mathfrak{h}) \neq 0$ implies $\mathcal{R}_1(\mathfrak{h}) \neq 0$ only under an assumption on the representation $\mathfrak{h} \subset \mathfrak{so}(n)$.

Proposition 2 *Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be an irreducible representation of real type of a simple subalgebra $\mathfrak{h}$ such that $\mathcal{P}_1(\mathfrak{h}) \neq 0$. If the irreducible representation $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ is given by the Dynkin diagram with only 1 or 2 non-zero labels, then $\mathcal{R}_1(\mathfrak{h}) \neq 0$, i.e. $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a symmetric Riemannian space.*

Proof. If the label is only one, then $\mathbb{R}^n \otimes \mathfrak{h}$ contains exactly one submodule isomorphic to $\mathbb{R}^n$, which is equal to $\tau(\mathbb{R}^n \otimes \mathrm{id}_\mathfrak{h})$, this implies the proof.

Suppose that there are two non-zero labels. Then $\mathbb{R}^n \otimes \mathfrak{h}$ contains exactly two submodules isomorphic to $\mathbb{R}^n$, one of them is equal to $\tau(\mathbb{R}^n \otimes \mathrm{id}_\mathfrak{h})$. The other one is contained in the subspace $(\mathbb{R}^n \otimes \mathfrak{h})_0 \subset \mathbb{R}^n \otimes \mathfrak{h}$ consisting of linear maps $\varphi : \mathbb{R}^n \to \mathfrak{h}$ with $\widetilde{\mathrm{Ric}}(\varphi) = 0$ [6]. It is obvious that the projection of $\mathcal{P}_1(\mathfrak{h})$ to $\tau(\mathbb{R}^n \otimes \mathrm{id}_\mathfrak{h})$ is not empty. Clearly, the subspace

$W\subset\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathfrak{h}$ of elements annihilated by $\mathfrak{h}$ is two-dimensional; it contains the subspace $\mathbb{R}\,\mathrm{id}_{\mathfrak{h}}\subset\odot^2\mathfrak{h}\subset\wedge^2\mathbb{R}^n\otimes\mathfrak{h}$. Since $\mathcal{P}_1(\mathfrak{h})\simeq\mathbb{R}^n$, there exists an $\mathfrak{h}$-equivariant isomorphism $S:\mathbb{R}^n\to\mathcal{P}_1(\mathfrak{h})$, $S\in W$. If $W\subset\wedge^2\mathbb{R}^n\otimes\mathfrak{h}$, then $S\in\mathcal{R}_1(\mathfrak{h})$. Otherwise, $W=\mathbb{R}\,\mathrm{id}_{\mathfrak{h}}\oplus\mathbb{R}\psi$, where $\psi\in\odot^2\mathbb{R}^n\otimes\mathfrak{h}$. Since $\mathcal{P}_1(\mathfrak{h})\not\subset(\mathbb{R}^n\otimes\mathfrak{h})_0$, $S\notin\mathbb{R}\psi$. The element $T\in\wedge^2\mathbb{R}^n\otimes\mathfrak{h}$ defined in the above lemma belongs to W, hence $T=c\,\mathrm{id}_{\mathfrak{h}}$ for some non-zero $c\in\mathbb{R}$. Next, $\mathrm{id}^*_{\mathfrak{h}}=\mathrm{id}_{\mathfrak{h}}$, and from the lemma it follows that $\mathrm{id}_{\mathfrak{h}}\in\mathcal{R}_1(\mathfrak{h})$. This proves the proposition. $\Box$

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